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REPRODUCING KERNEL FUNCTIONS FOR THE

SARD CORNER SPACES $B_{[p,q]}$

RICHARD FRANKE

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

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Provost

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The reproducing kernel functions for the Sard corner spaces $B_{[p,q]}$ are integrated to obtain explicit expressions. The smoothness properties of these functions are discussed.

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1. Introduction

Reproducing kernel functions arise naturally in approximation theory in Hilbert spaces, and are a useful theoretical tool, and can also be used for computations. Since I have used reproducing kernels in Sard corner spaces in my own work, I have felt it would be useful to have the formulas (given in terms of integrals) evaluated and written out in a more explicit form. Therefore this small report was written to discuss Sard corner spaces and to develop those formulas.

2. Sard Spaces

Sard spaces of type B are spaces of functions which have a certain type of Taylor series expansion with remainder [2]. The partial derivatives which exist, and in terms of which the expansion is given, is specified by the "complete core". For details the reference should be consulted. We follow the notation of Sard so that subscripts denote differentiation. The complete core of $B_{\ulcorner p,q \urcorner}$ (" B corner p,q ") is made up of the elements

$$\omega_{s,t} = \{f_{p,q}(s,t)\}$$

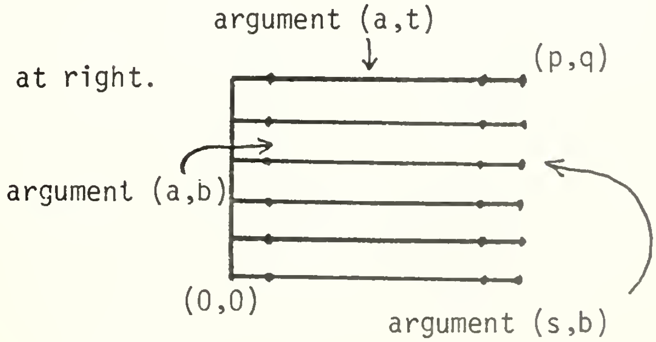
$$\omega_{s,b} = \{f_{p,j}(s,b): \; j < q\}$$

$$\omega_{a,t} = \{f_{i,q}(a,s): \; i < p\}$$

$$\omega_{a,b} = \{f_{i,j}(a,b): \; i < p, \; j < q\}$$

Here (a,b) is a fixed point.

The differentiation diagram is at right.



A function $f \in B_{\lceil p,q \rceil}$ can be expressed as

$$f(s,t) = \sum_{\substack{i<p\\ j<q}} (s-a)^{(i)}(t-b)^{(j)} f_{i,j}(a,b)$$

$$+ \sum_{i<p} (s-a)^{(i)} \int_b^t (t-\tilde{t})^{(q-1)} f_{i,q}(a,\tilde{t})d\tilde{t}$$

$$+ \sum_{j<q} (t-b)^{(j)} \int_a^s (s-\tilde{s})^{(p-1)} f_{p,j}(\tilde{s},b)d\tilde{s}$$

$$+ \int_a^s \int_b^t (s-\tilde{s})^{(p-1)}(t-\tilde{t})^{(q-1)} f_{p,q}(\tilde{s},\tilde{t})d\tilde{t}d\tilde{s}$$

The inner product for $B_{\lceil p,q \rceil}$, as defined by Barnhill and Nielson [1], is

$$(f,g) = \sum_{\substack{i<p\\ j<q}} f_{ij}(a,b)g_{i,j}(a,b)$$

$$+ \sum_{i<p} \int_\beta^{\tilde{\beta}} f_{i,q}(a,\tilde{t})g_{i,q}(a,\tilde{t})d\tilde{t}$$

$$+ \sum_{j<q} \int_\alpha^{\tilde{\alpha}} f_{p,j}(\tilde{s},b)g_{p,j}(\tilde{s},b)d\tilde{s}$$

$$\int_\alpha^{\tilde{\alpha}} \int_\beta^{\tilde{\beta}} f_{p,q}(\tilde{s},\tilde{t})g_{p,q}(\tilde{s},\tilde{t})d\tilde{t}d\tilde{s}$$

Here $[\alpha,\tilde{\alpha}] \times [\beta,\tilde{\beta}]$ is the rectangle which contains the region of interest.

The above leads to the reproducing kernel function, which in turn will allow construction of the representers of functionals, particularly point evaluation functionals, in which we are interested. The reproducing kernel function is

$$(*)\ K(u,v;s,t) = \sum_{\substack{i<p\\ j<q}} (u-a)^{(i)}(v-b)^{(j)}(s-a)^{(i)}(t-b)^{(j)}$$

$$+ \sum_{i<p} (u-a)^{(i)}(s-a)^{(i)} \int_b^v (v-\tilde{t})^{(q-1)}(t-\tilde{t})^{(q-1)} \psi(b,\tilde{t},t)d\tilde{t}$$

$$+ \sum_{j<q} (v - b)^{(j)} (t - b)^{(j)} \int_a^u (u - \tilde{s})^{(p-1)} (s - \tilde{s})^{(p-1)} \psi(a,\tilde{s},s) d\tilde{s}$$

$$+ \int_a^u \int_b^v (u - \tilde{s})^{(p-1)} (s - \tilde{s})^{(p-1)} (v - \tilde{t})^{(q-1)} (t - \tilde{t})^{(q-1)} \psi(b,\tilde{t},t) \psi(a,\tilde{s},s) d\tilde{t} d\tilde{s}$$

In the above the function ψ is defined as

$$\psi(a,\tilde{s},s) \quad \begin{cases} 1 & \text{if } a \le \tilde{s} < s \\ -1 & \text{if } s \le \tilde{s} < a \\ 0 & \text{otherwise} \end{cases}$$

For the above case, the kernel function can be factored into $K(u,v;s,t) = g_p(a;u,s)\, g_q(b;v,t)$, where

$$g_p(a;u,s) = \sum_{i<p} (u - a)^{(i)} (s - a)^{(i)} + \int_a^u (u - \tilde{s})^{(p-1)} (s - \tilde{s})^{(p-1)} \psi(a,\tilde{s},s) d\tilde{s}$$

This fact simplifies investigation of the properties of the representers of functionals considerably. It is desirable to obtain an expression for $g_p(a;u,s)$ which does not involve the integral. Repeated integration by parts, and assuming that $a \le u,s$ yields

$$g_p(a;u,s) = (-1)^p (u - s)_+^{(2p-1)}$$

$$+ \sum_{i<p} \{(u - a)^{(i)} (s - a)^{(i)} + (-1)^i (s - a)^{(p-1-i)} (u - a)^{(p+i)}\} .$$

Note here that we have lost the obvious symmetry of g_p in u and s, although not the actual symmetry, of course. We can regain it however, at the expense of two formulas for g_p, depending on whether $u \le s$ or $s < u$. To do this expand $(u - s)_+^{(2p-1)}$ (for $u > s$) in a binomial way, obtaining

$$(-1)^p (u - s)^{(2p-1)} = \frac{(-1)^p}{(2p-1)!} \sum_{k=0}^{2p-1} (2p-1)! (-1)^{2p-1-k} \frac{(u-a)^k}{k!} \frac{(s-a)^{2p-1-k}}{(2p-1-k)!}$$

$$= (-1)^p \sum_{k=0}^{2p-1} (-1)^{2p-1-k} (u - a)^{(k)} (s - a)^{(2p-1-k)}$$

$$= (-1)^p \sum_{k=0}^{p-1} (-1)^{2p-1-k}(u - a)^{(k)}(s - a)^{(2p-1-k)}$$

$$+ (-1)^p \sum_{k=p}^{2p-1} (-1)^{2p-1-k}(u - a)^{(k)}(s - a)^{(2p-1-k)}$$

$$= \sum_{i<p}(u - a)^{(p-1-i)}(s - a)^{(p+i)}(-1)^i \qquad \underline{(k \leftarrow p-i-1)}$$

$$- \sum_{i<p}(u - a)^{(p+i)}(s - a)^{(p-1-i)}(-1)^i \qquad \underline{(k \leftarrow p+i)}$$

Thus, $(-1)^p(u - s)^{(2p-1)} + \sum_{i<p}(u - a)^{(p+i)}(s - a)^{(p-1-i)}(-1)^i$

$$= \sum_{i<p}(u - a)^{(p-1-i)}(s - a)^{(p+1)}(-1)^i ,$$

and we see that if we interchange u and s the expression for g_p is unchanged, and thus we can take

(1) $g_p(a;u,s) = \sum_{i<p} \{(u - a)^{(i)}(s - a)^{(i)} + (-1)^i(u - a)^{(p+i)}(s - a)^{(p-1-i)}\}$

when $a \leq u < s$, and

(2) $g_p(a;u,s) = g_p(a;s,u)$ when

$a \leq s \leq u$. When $s < a \leq u$ we find that

(3) $g_p(a;u,s) = \sum_{i<p}(u - a)^{(i)}(s - a)^{(i)}$

For $u < a$ it is easily observed that the only difference is that the function $\psi(a,\tilde{s},s)$ changes sign, hence the terms from the integral appear with opposite sign, so we can say

(1') $g_p(a;u,s) = \sum_{i<p} \{(u - a)^{(i)}(s - a)^{(i)} - (-1)^i(u - a)^{(p+1)}(s - a)^{(p-1-i)}\}$

when $s \leq u < a$, and

(2') $g_p(a;u,s) = g_p(a;s,u)$ when $u < s < a$. When $u \leq a \leq s$ we find that

(3') $g_p(a;u,s) = \sum_{i<p}(u - a)^{(i)}(s - a)^{(i)}$.

It is possible to write the reproducing kernel function in a somewhat more concise manner by defining the function

$$G_p(a;u,s) = (-1)^p (s-u)_+^{(2p-1)}$$

$$+ \sum_{i<p} \{(u-a)^{(i)}(s-a)^{(i)} + (-1)^i (u-a)^{(p-1-i)}(s-a)_+^{(p+i)}\} ,$$

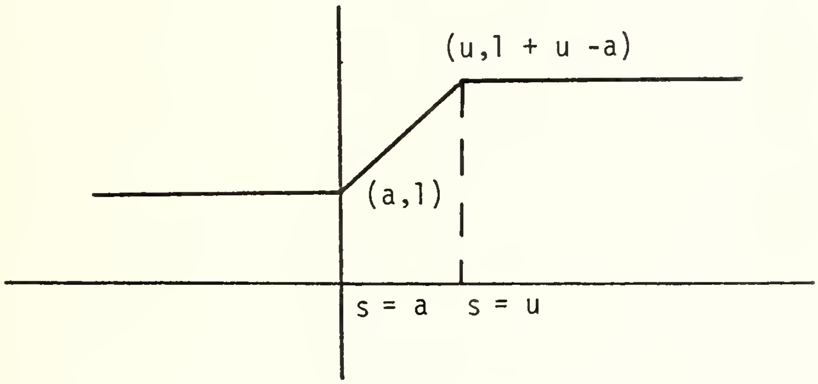
for $a \le u$, and then observing that for cases 1, 2, and 3

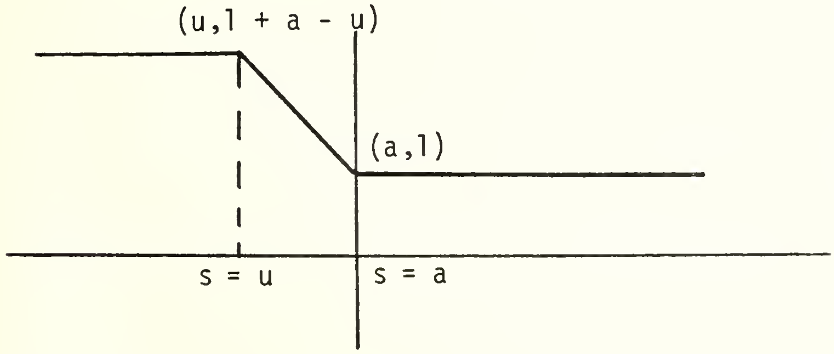
$$g_p(a;u,s) = G_p(a;u,s) ,$$

while for cases 1', 2', and 3' ,

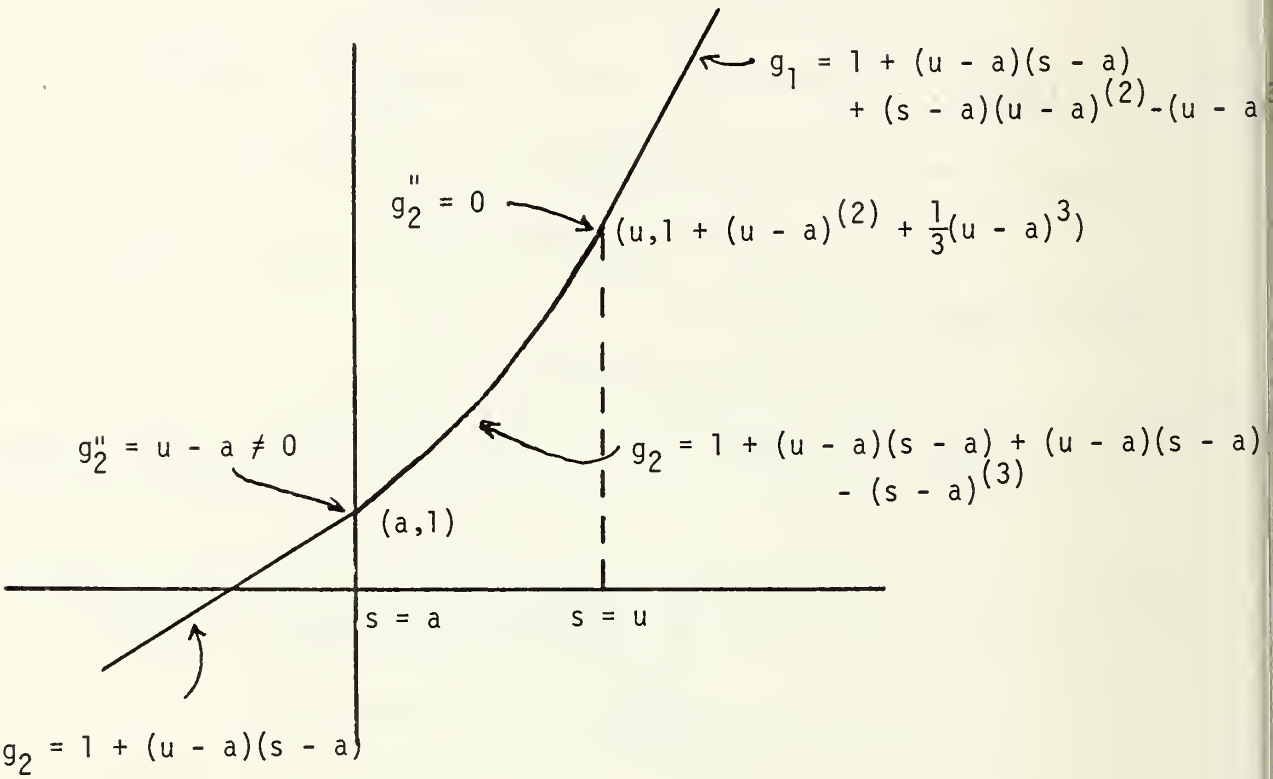
$$g_p(a;u,s) = G_p(-a;-u,-s) .$$

For $p = 1$ and a fixed value of u , graphs of $g_1(a;u,s)$ are given below.





For $p = 2$, the graph of the function $g_2(a;u,s)$ is given below for $a < u$. For $a > u$, the graph is flipped about $s = a$, as in the $p = 1$ case.



We note that g_2 is C^2 in s except at $s = a$, where it has a continuous first derivative, but not a continuous second derivative. In general g_p is C^{2p-2} in s, except at $s = a$, where it is C^{p-1}.

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2. A. Sard, Linear Approximation, Mathematical Surveys, No. 9, American Mathematical Society, Providence, RI, 1963

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